

CONTINUA AND VARIOUS TYPES OF HOMOGENEITY⁽¹⁾

BY

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1. **Definitions.** A point set M is said to be *n-homogeneous* if for any n points $x_1, x_2, \dots, x_n$ of M and any n points $y_1, y_2, \dots, y_n$ of M there is a homeomorphism of M onto itself that carries $x_1 + x_2 + \dots + x_n$ onto $y_1 + y_2 + \dots + y_n$. If there is such a homeomorphism which carries x_i into y_i ($i \leq n$), then M is said to be *strongly n-homogeneous*. A point set M is said to be *nearly n-homogeneous* if for any n points $x_1, x_2, \dots, x_n$ of M and any n open subsets⁽²⁾ $D_1, D_2, \dots, D_n$ of M there exist n points $y_1, y_2, \dots, y_n$ of $D_1, D_2, \dots, D_n$, respectively, and a homeomorphism of M onto itself that carries $x_1 + x_2 + \dots + x_n$ onto $y_1 + y_2 + \dots + y_n$. For convenience throughout this paper, the terms "*n-homogeneous*" and "*nearly n-homogeneous*" are used only where $n > 1$. Where $n = 1$, the terms "*homogeneous*" and "*nearly homogeneous*" are used. Continua possessing these types of homogeneity have been investigated previously in [4; 6].

A continuum M is said to be *ap syndetic at the point x of M* if for any point y of $M - x$ there is a subcontinuum K of M and an open subset U of M such that $M - y \supset K \supset U \supset x$. The continuum M is said to be *ap syndetic* if it is ap syndetic at each of its points.

A subset H of the connected point set M is said to *separate M* if $M - H$ is not connected.

A point x of a continuum M is said to be a *cut point* of M if there exist two points y and z in $M - x$ such that every subcontinuum of M that contains $y + z$ also contains x .

2. **Homogeneous plane continua.** Let K denote a nondegenerate homogeneous bounded plane continuum. F. B. Jones [9] has shown that K is a simple closed curve provided it either is ap syndetic or contains no cut point, and H. J. Cohen [7] has shown that K is a simple closed curve if it either contains a simple closed curve or is arc-wise connected. These results generalized Mazurkiewicz's theorem that K is a simple closed curve if it is locally connected [13]. Similar results are obtained here for a bounded plane continuum which is nearly homogeneous and separates the plane into a finite number of connected domains. It can easily be seen that it is necessary in

Presented to the Society, November 27, 1954; received by the editors June 25, 1956 and in revised form, October 18, 1956 and December 21, 1956.

(¹) Some of this work was supported by the National Science Foundation under G-2574.

(²) A subset of a continuum M that is open relative to M is called an *open subset* of M .

Theorem 1 to require that M separate the plane as there exists a dendron which is nearly homogeneous. Also, a locally connected bounded plane continuum described by Sierpinski [17] is nearly homogeneous and has infinitely many complementary domains. It was shown in [4] that a bounded plane continuum M is the boundary of each of its complementary domains provided M is nearly homogeneous and does not have infinitely many complementary domains. Sierpinski's example cited above shows that it is necessary to require that M should not have infinitely many complementary domains, and it is shown here (Theorem 2) that this requirement is not necessary if M is homogeneous. Results more complete than those in Theorems 3, 4, and 5 have been obtained by F. B. Jones [11] for homogeneous continua. A more complete bibliography and a history of results on homogeneous plane continua will appear in a paper by Bing and Jones [2].

THEOREM 1. *If the bounded plane continuum M is nearly homogeneous and separates the plane into a finite number of connected domains, then M is a simple closed curve provided any one of the following four requirements is fulfilled:*

- (a) M contains a simple closed curve.
- (b) M is aposyndetic.
- (c) M contains no cut point.
- (d) M is arc-wise connected.

Proof of (a). It follows from [4, Theorem 7] that no proper subcontinuum of M separates the plane. Hence M is a simple closed curve.

Proof of (b). No proper subcontinuum of M separates the plane [4, Theorem 7], and hence it follows from a theorem proved by R. L. Moore [14, Theorem 2] that no subcontinuum of M separates M . Bing [1, Theorem 2] has shown that any such aposyndetic continuum is locally connected. That M is a simple closed curve follows from (a) and the fact that every locally connected bounded plane continuum which separates the plane contains a simple closed curve.

Proof of (c). As in the proof of (b), no proper subcontinuum of M separates M . Bing [1, Theorem 10] has shown that such a continuum is a simple closed curve if it is not cut by any one of its points.

Proof of (d). It follows from [4, Theorem 7] that no proper subcontinuum of M separates the plane. Since M is decomposable, it follows from a theorem proved by Kuratowski [12, Theorem 5] that M is the sum of two continua M_1 and M_2 irreducible between the same pair of points. Hence $M_1 \cdot M_2$ is the sum of two mutually separated sets H_1 and H_2 , and there is an arc K in M irreducible from H_1 to H_2 and lying in one of the sets M_1 and M_2 . Consider the case in which K is a subset of M_1 . Since $K \cdot M_2$ is not connected, it follows from a well known theorem (Janiszewski) that $K + M_2$ separates the plane. Hence $K + M_2 = M$ and $M_1 - M_1 \cdot M_2 = K - K \cdot M_2$. This implies that M is aposyndetic at each point of the set $M_1 - M_1 \cdot M_2$, and since this set is open

relative to M , it follows from the near-homogeneity of M that M is aposynthetic. Hence it follows from (b) that M is a simple closed curve.

THEOREM 2. *Every homogeneous bounded plane continuum is the boundary of each of its complementary domains.*

Proof. Suppose that the boundary K of some complementary domain of a homogeneous bounded plane continuum M is different from M . It is well known that K is a proper subcontinuum of M and it is easy to see that K separates the plane. From a classification of homogeneous decomposable bounded plane continua given by F. B. Jones [11, Theorem 2], it follows that M is indecomposable. Since any homeomorphic image of K separates the plane (Brouwer), it follows from the homogeneity of M that every composant⁽³⁾ of M contains a continuum that separates the plane. But no indecomposable continuum is separated by one of its subcontinua. Hence each composant of M contains a continuum that is the boundary of some complementary domain of M . Since M has uncountably many mutually exclusive composants, this leads to the contradiction that M has uncountably many complementary domains.

THEOREM 3. *If the compact metric continuum M is nearly homogeneous and H is an indecomposable proper subcontinuum of M , then $\text{cl}(M-H)$ intersects every composant of H .*

Proof. Suppose some composant L of H does not intersect $\text{cl}(M-H)$. Let R be an open subset of M intersecting L but not $\text{cl}(M-H)$. For each point y of M there is a point x of R and a homeomorphism of M onto itself that carries x into y . Hence for each point y of M , there is an indecomposable subcontinuum K of M and an open subset D of M such that $K \supset D \supset y$ and some composant of K does not intersect $\text{cl}(M-K)$. Since M can be covered by a finite number of open sets such as D , it follows that there is a least integer n such that M is the sum of n indecomposable continua $M_1, M_2, \dots, M_n$ such that no one of them is a subset of the sum of the others and for each i ($i \leq n$), some composant of M_i does not intersect $\text{cl}(M-M_i)$.

Now suppose that some composant K_1 of M_1 intersects M_2 and does not intersect $\text{cl}(M-M_1)$. Since M_2 intersects $\text{cl}(M-M_1)$, it follows that some subcontinuum Z of M_2 is irreducible from $\text{cl}(M-M_1)$ to some point of K_1 . From the fact that $Z = \text{cl}(Z - \text{cl}(M-M_1))$ [15, Theorem 37, p. 22], it follows that Z is a proper subcontinuum of M_1 intersecting both K_1 and $\text{cl}(M-M_1)$. This involves the contradiction that K_1 intersects $\text{cl}(M-M_1)$. Hence it follows that some composant of M_1 does not intersect M_2 , and similarly, for each j ($2 \leq j \leq n$), some composant of M_1 does not intersect M_j . By [5, Theorem 2], some composant of M_1 does not intersect $M_2 + M_3 + \dots + M_n$. In a

⁽³⁾ If x is a point of a continuum M , the sum of all of the proper subcontinua of M that contain x is called a *composant* of M .

similar manner it can be shown that for each i ($i \leq n$), some composant of M_i does not intersect $M_1 + M_2 + \cdots + M_{i-1} + M_{i+1} + \cdots + M_n$. By [3, Theorem 1], M is n -indecomposable⁽⁴⁾, and from the requirement that H is a proper subcontinuum of M it readily follows that $n > 1$. But this is impossible for a compact metric continuum which is nearly homogeneous [4, Theorem 2]. That $\text{cl}(M-H)$ intersects every composant of H follows from this contradiction.

THEOREM 4. *If the bounded plane continuum M is nearly homogeneous and H is an indecomposable proper subcontinuum of M such that $M-H$ has only a finite number of components, then $\text{cl}(M-H) = M$.*

Proof. Suppose that $\text{cl}(M-H)$ is a proper subset of M . It has been shown by Rutt [16, Lemma 1] that no set which is the closure of a component of $M-H$ intersects every composant of M . Hence by [5, Theorem 2], some composant of H does not intersect $\text{cl}(M-H)$. This is contrary to Theorem 3.

THEOREM 5. *If the bounded plane continuum M is nearly homogeneous and separates the plane into a finite number of connected domains, then every indecomposable proper subcontinuum of M is a continuum of condensation of M .*

Proof. It has been shown that M is the boundary of each of its complementary domains [4, Theorem 7]. Since such a continuum is not separated by any one of its subcontinua [14, Theorem 2], the conclusion of this theorem then follows from Theorem 4.

3. Separation properties and homogeneity. It has been shown previously that a decomposable plane continuum is locally connected if it is nearly n -homogeneous [4; 6]. By imposing additional conditions in the hypotheses, some similar results are obtained here for compact metric continua.

THEOREM 6. *If the decomposable metric continuum M is nearly 2-homogeneous, then M is aposyndetic.*

Proof. Let M_1 and M_2 be two proper subcontinua of M such that $M_1 + M_2 = M$, and let x_1 and x_2 be any two points of M . There exist two points y_1 and y_2 in $M-M_1$ and $M-M_2$ respectively and a homeomorphism f of M onto itself such that $f(y_1 + y_2) = x_1 + x_2$. Then $f(M_1) + f(M_2) = M$ and neither of the continua $f(M_1)$ and $f(M_2)$ contains both x_1 and x_2 . Hence M is aposyndetic at x_1 with respect to x_2 , and from this it follows that M is aposyndetic.

THEOREM 7. *Every 2-homogeneous compact metric continuum is aposyndetic.*

⁽⁴⁾ A continuum M is said to be n -indecomposable if M is the sum of n continua such that no one of them is a subset of the sum of the others and M is not the sum of $n+1$ such continua. In some of the references such continua are said to be *indecomposable under index n* .

Proof. This theorem follows from Theorem 6 and the fact that every 2-homogeneous compact metric continuum is decomposable. (See proof of [4, Theorem 10].)

REMARK. Using the fact that a decomposable bounded plane continuum is locally connected if it is nearly 2-homogeneous [4, Theorem 9], I proved that every 2-homogeneous bounded plane continuum is a simple closed curve [4, Theorem 10]. I wish to observe here that this result is a direct consequence of Theorem 7 and F. B. Jones' theorem that every homogeneous aposyndetic bounded plane continuum is a simple closed curve [9]. In fact, it follows that a decomposable bounded plane continuum is a simple closed curve if it is homogeneous and nearly 2-homogeneous. This result is also included in a more recent result by F. B. Jones [11, Theorem 2].

THEOREM 8. *If the compact metric continuum M is aposyndetic and hereditarily unicoherent⁽⁶⁾, then M is locally connected.*

Proof. Let x be a point of M and let K be a subcontinuum of M not containing x . Jones [8, Theorem 7] has shown that every aposyndetic continuum is freely decomposable⁽⁶⁾. Hence for each point y of K , M is the sum of two continua X and Y such that x and y lie in $M - Y$ and $M - X$ respectively. Consider the collection of all such pairs of continua. Then K is covered by the collection of all open sets such as $M - X$, and hence K is covered by a finite number of such open sets. There exists a positive integer m and continua $X_1, X_2, \dots, X_m, Y_1, Y_2, \dots, Y_m$ such that $K \subset Y_1 + Y_2 + \dots + Y_m$ and for each i ($i \leq m$), $X_i + Y_i = M$ and $x \in M - Y_i$. Since M is hereditarily unicoherent, it follows that the common part X' of the continua $X_1, X_2, \dots, X_m$ is a continuum that does not intersect K . Clearly $M = X' + Y_1 + Y_2 + \dots + Y_m$. Hence it has been shown that for any point x of M and any subcontinuum K of M not containing x , the continuum M is the sum of a finite number of continua no one of which both intersects K and contains x . This condition is sufficient for a compact metric continuum to be locally connected [15, Theorem 51, p. 134]. (Also, see [8, Theorem 8].)

COROLLARY 8.1. *If the decomposable compact metric continuum M is hereditarily unicoherent and nearly 2-homogeneous, then M is locally connected⁽⁷⁾.*

THEOREM 9. *If the decomposable compact metric continuum M is unicoherent and nearly homogeneous, then for every positive integer n some subcontinuum of M separates M into more than n components.*

⁽⁶⁾ A continuum M is said to be *unicoherent* if every two continua whose sum is M have a connected intersection. A continuum is said to be *hereditarily unicoherent* if each of its subcontinua is unicoherent.

⁽⁶⁾ A continuum M is said to be *freely decomposable* if for any two points x and y of M there exist two continua whose sum is M such that neither of them contains both x and y .

⁽⁷⁾ F. B. Jones [10] has shown that there does not exist a homogeneous continuum which satisfies the hypothesis of this corollary.

Proof. Suppose the contrary. Since M is decomposable and unicoherent, it follows that some subcontinuum of M separates M . Hence it follows from the above supposition that for some integer k greater than one some subcontinuum of M separates M into k components and no subcontinuum of M separates M into more than k components. Sorgenfry [18, Theorem 3.6] has shown that such a continuum is irreducible about some k of its points. This involves a contradiction since no decomposable compact metric continuum is both nearly homogeneous and irreducible about some finite set [4, Theorem 4].

COROLLARY 9.1. *If the nondegenerate compact metric continuum M is unicoherent and n -homogeneous, then for every positive integer k some subcontinuum of M separates M into more than k components.*

COROLLARY 9.2. *If the unicoherent compact metric continuum M is n -homogeneous and H is a subset of M consisting of n points, then there exists a subcontinuum of M which separates each two points of H from each other in M .*

4. Some characterizations of a simple closed curve. Simple closed curves in the plane have been characterized with some type of homogeneity requirement by Mazurkiewicz [13], Jones [9], Cohen [7], and the author [4]. (Also, see Theorem 1 in this paper.) A decomposable compact metric continuum that is continuum-wise homogeneous has been characterized as a simple closed curve [6]. The proofs of the characterizations given here rely heavily upon Bing's characterization of a simple closed curve as a compact metric continuum that is neither cut by any one of its points nor separated by any one of its subcontinua [1] and upon Whyburn's theorem that all except a countable number of local separating points of a compact metric continuum are points of order two in that continuum [19].

THEOREM 10. *If the decomposable compact metric continuum M is nearly 2-homogeneous and there exist two points of M such that no subcontinuum of M separates them in M , then M is a simple closed curve.*

Proof. Let x_1 and x_2 denote two points of M such that no subcontinuum of M separates them in M . Suppose that some subcontinuum K of M separates M . Then $M - K$ is the sum of two mutually separated sets D_1 and D_2 that are open relative to M . There exist a point y_1 in D_1 and a point y_2 in D_2 such that there is a homeomorphism of M onto itself that carries $x_1 + x_2$ onto $y_1 + y_2$. This involves the contradiction that the inverse of this homeomorphism carries the continuum K onto a continuum that separates x_1 from x_2 . Hence no subcontinuum of M separates M .

By Theorem 6, M is aposyndetic. Jones [8, Theorems 3 and 4] has shown that the class of aposyndetic compact metric continua is equivalent to the class of semi-locally-connected compact metric continua, and Whyburn [21, 6.21] has shown that every nonseparating point of a semi-locally-connected

compact metric continuum is also a noncut point of that continuum. Since no point separates M , it follows that no point cuts M . Bing [1, Theorem 10] has shown that a compact metric continuum is a simple closed curve if it is neither cut by any point nor separated by any one of its subcontinua. Hence M is a simple closed curve.

COROLLARY 10.1. *If the nondegenerate compact metric continuum M is 2-homogeneous and is not separated by any one of its subcontinua, then M is a simple closed curve.*

COROLLARY 10.2. *If the nondegenerate compact metric continuum M is strongly 4-homogeneous, then some subcontinuum of M separates M .*

Proof. This corollary follows from Corollary 10.1 and the fact that no simple closed curve is strongly 4-homogeneous.

THEOREM 11. *If the nondegenerate compact metric continuum M is multicoherent⁽⁸⁾ and 2-homogeneous, then M is a simple closed curve.*

Proof. This theorem follows from Corollary 10.1 and the fact that no multicoherent compact metric continuum is separated by one of its subcontinua.

THEOREM 12. *If M is a compact metric continuum such that for any proper subcontinuum H of M and any two sets K_1 and K_2 in $M - H$ each consisting of two points there is a homeomorphism of M onto itself that carries H onto itself and K_1 onto K_2 , then M is a simple closed curve.*

Proof. Suppose that M is not a simple closed curve. By Corollary 10.1, some subcontinuum H of M separates M . Let x_1 and x_2 be two points lying in the same component of $M - H$ and let y_1 and y_2 be two points lying in different components of $M - H$. A contradiction results since there is no homeomorphism of M onto itself that carries H onto itself and $x_1 + x_2$ onto $y_1 + y_2$.

THEOREM 13. *If the homogeneous compact metric continuum M is separated by some countable set, then M is a simple closed curve.*

Proof. Any countable set that separates M contains a local separating point of M [19, Corollary 5a]. Since M is homogeneous, every point of M is a local separating point of M . Hence M contains a point of order 2 [19, Theorem 9]. The homogeneity of M thus requires that every point of M is of order 2, and hence M is a simple closed curve (Menger).

THEOREM 14. *If the nondegenerate compact metric continuum M is homogeneous and hereditarily locally connected, then M is a simple closed curve.*

Proof. Every hereditarily locally connected compact metric continuum is

⁽⁸⁾ A continuum M is said to be *multicoherent* if there do not exist two proper subcontinua M_1 and M_2 of M such that $M_1 + M_2 = M$ and $M_1 \cdot M_2$ is connected.

a rational curve [20], and hence some countable set separates M . That M is a simple closed curve follows from Theorem 13.

5. Homogeneous triodic continua. A continuum is said to be a triod if it is the sum of three continua such that their intersection is a proper subcontinuum of each of them and is the intersection of each two of them.

THEOREM 15. *If the decomposable compact metric continuum M is nearly 2-homogeneous and is not a triod, then M is locally connected.*

Proof. The near 2-homogeneity and the decomposability of M imply that M is freely decomposable, and Sorgenfry [18, Theorem 3.1] has shown that a freely decomposable compact metric continuum is locally connected if it is not a triod.

COROLLARY 15.1. *If the compact metric continuum M is 2-homogeneous and is not a triod, then M is locally connected.*

QUESTION. Is every 2-homogeneous compact metric continuum locally connected?

THEOREM 16. *If the decomposable compact metric continuum M is unicoherent and nearly homogeneous, then M is a triod.*

Proof. Suppose M is not a triod. Sorgenfry [18, Theorem 3.2] has shown that such a unicoherent continuum is irreducible between some two points. This involves a contradiction as no decomposable compact metric continuum is both nearly homogeneous and irreducible about a finite set [4, Theorem 4].

COROLLARY 16.1. *If the nondegenerate compact metric continuum M is n -homogeneous and unicoherent, then M is a triod.*

COROLLARY 16.2. *If the decomposable bounded plane continuum M does not separate the plane and is nearly homogeneous, then M is a triod.*

Added in proof. The theorems in this paper that apply to continua that are 2-homogeneous can be strengthened to include continua that are n -homogeneous, where $n > 2$, by using Morton Brown's result that every n -homogeneous continuum is $(n-1)$ -homogeneous [*n -Homogeneity implies $(n-1)$ -homogeneity*, Bull. Amer. Math. Soc. Abstract 60-4-448].

I have recently noticed that a locally connected compact metric continuum is a simple closed curve provided it is nearly homogeneous and is not a triod. Hence it can be concluded in Theorem 15 and Corollary 15.1 that M is a simple closed curve. A note on this will be submitted in another paper.

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